OPERATOR CHARACTERIZATIONS OF $\mathscr{L}_p$ -SPACES

BY

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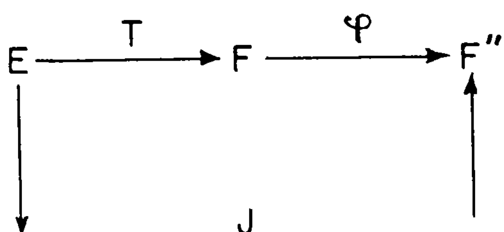
ABSTRACT

We present a survey, with brief indications of proof, of recent results characterizing the $\mathscr{L}_p$-spaces of Lindenstrauss and Pelczynski in terms of the behavior of various classes of bounded linear operators.

In the following, E and F denote Banach spaces and $\mathscr{L}(E,F)$ denotes the class of all bounded linear operators from E to F. We will also need the following classes of operators from E to F: An operator from E to F is *p-absolutely summing*, denoted $T \in \Pi_p(E,F)$, if there is a constant $C > 0$ so that for arbitrary $x_1, \cdots, x_n$ in E,

$$[\sum_{i=1}^{n} \|Tx_i\|^p]^{1/p} \leqq C \sup\{[\sum_{i=1}^{n} |\langle x_i, f\rangle|^p]^{1/p} : \|f\| \leqq 1\};$$

p-integral, denoted $T \in I_p(E,F)$, if T admits the following factorization:



where μ is a probability measure, J the canonical injection of $L_\infty(\mu)$ into $L_p(\mu)$ and ϕ the canonical operator for F into F''; and, *nuclear*, denoted $T \in N(E,F)$, if there are sequences (f_i) in E' and (y_i) in F such that

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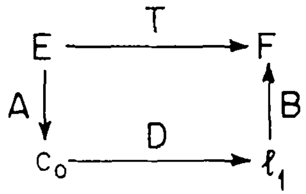
$$T = \sum_{i=1}^{\infty} f_i \otimes y_i \quad \text{and} \quad \sum_{i=1}^{\infty} \|f_i\| \, \|y_i\| < +\infty.$$

Here $f_i \otimes y_i$ is the rank one operator given by $x \to f_i(x)y_i$.

It is immediate from the definition that a nuclear operator T may be factored as follows:



where D is a diagonal nuclear operator and A and B are compact.

Finally, an operator T from E to F is *quasi-nuclear*, written $T \in QN(E,F)$, if there is a sequence (f_n) in E' with $\sum_{n=1}^{\infty} \|f_n\| < +\infty$ such that

$$\|Tx\| \leqq \sum_{n=1}^{\infty} |\langle x, f_n \rangle|$$

for each $x \in E$.

The p-absolutely summing operators have been studied by Grothendieck [4, 5], Saphar [27], Pietsch [23] and others. The p-integral operators have been studied by Grothendieck [4,5], Pietsch and Persson [21,22] and Pelczynski [20]. The nuclear operators were introduced by Grothendieck [4] and the quasi-nuclear operators by Pietsch [24]. Pietsch showed that if $T \in QN(E,F)$ and F is imbedded in some $l_\infty(\Gamma)$ with imbedding α then $\alpha T \in N(E, l_\infty(\Gamma))$. One always has the following inclusions between the classes of operators defined above:

$$N \subset I_1 \subset \Pi_1 \quad \text{and} \quad QN \subset \Pi_1.$$

Before proceeding to the theorems, we recall a (still unsolved) conjecture of Grothendieck [4] which motivated much of what follows: *If* $\mathscr{L}(E,F) = N(E,F)$ *then one of E or F must be finite dimensional.*

While it is clear that the nuclear operators are the easiest operators to obtain after the operators of finite rank, in some respects they are difficult to work with. One of the difficulties is that the (y_i) occurring in the definition of a nuclear operator have, in general, nothing to do with the range, $T(E)$, of the nuclear operator from E to F. With this in mind, the author and his colleague Stegall introduced the concept of a fully nuclear operator. An operator T from E to F is

$$\textit{fully nuclear}, \text{ written } T \in FN(E,F),$$

if the astricted operator $T_a: E \to \mathrm{cl}T(E)$ is nuclear ($\mathrm{cl}A$ denotes the closure of a set A). We were then able to prove the following [28]; *If* $\mathscr{L}(E,F) = FN(E,F)$ *then one of* E,F *must be finite dimensional.*

While this is at present the best result concerning the Grothendieck conjecture, it leaves much to be desired. Indeed, *the fully nuclear operators do not in general form a linear subspace.*

To see this, we first observe that the existence of nuclear, non-fully nuclear operators was essentially known to Grothendieck. Indeed, the existence of such operators is immediate from the following result of Grothendieck [4]: *Let* F *be a Banach space and* E *a closed subspace of* F *such that* E *is complemented in* E''. *Then the cannoical operator* $J: E \hat{\otimes} E' \to F \hat{\otimes} E'$ *is an into isomorphism if and only if* E *is complemented in* F.

Thus examples of nuclear, non-fully nuclear operators may be given by choosing a Banach space F with the approximation property, and a closed, non-complemented, reflexive subspace E with the approximation property. Then there is an element T of $F \hat{\otimes} E'$ that is in the closure of $E \hat{\otimes} E'$ but not in $E \hat{\otimes} E'$. Regarding T as an operator, $T: E \to F$, then $T(E) \subset E$, T is nuclear but the astriction is not nuclear. (We remark that there is a constructive example of a nuclear. operator which is not fully nuclear [13].) We now refer the reader to [28, p.473] for the pathological structure of $FN(E,F)$.

It is clear that if $\mathrm{cl}T(E)$ is complemented in F and $T \in N(E,F)$, then $T \in FN(E,F)$. Thus, in particular, for any Hilbert space H and any Banach space E,

$$\text{(A)} \qquad N(E,H) = FN(E,H).$$

Also, from the properties of the unit vector basis of c_0, it is clear that $\Pi_1(c_0, F) = N(c_0,F)$ for any Banach space F. Since the notion of an absolutely summing operator is independent of the range, it is clear that

$$\text{(B)} \qquad N(c_0,F) = FN(c_0,F)$$

for any Banach space F. Thus, in some cases, the fully nuclear operators are nicely behaved. This leads to the natural question: " What are the domain [range] spaces E for which $FN(E,F) = N(E,F)$ $[FN(F,E) = N(F, E)]$ for all Banach spaces F?" It is possible to give a complete answer to this question. For this we need the notion of an $\mathscr{L}_p$-space in the sense of Lindenstrauss and Pelczynski [16], which we now define.

For isomorphic Banach spaces E and F, the Banach-Mazur distance, $d(E,F)$, is defined by

$$d(E,F) = \inf \| T \| \| T^{-1} \|$$

the infimum taken over all isomorphisms of E onto F.

A Banach space E is an $\mathscr{L}_{p,\lambda}$-space if for every finite dimensional subspace A in E there is a finite dimensional subspace B of E with $A \subset B$ such that

$$d(B, l_n^p) \leqq \lambda,$$

where $\dim B = n$. Finally, the space E is an $\mathscr{L}_p$-space if it is an $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqq 1$. The $\mathscr{L}_p$-spaces generalize and include the classical $L_p(\Omega, \Sigma, \mu)$, $1 \leqq p \leqq \infty$ and $C(K)$-spaces. We will need the fact that the $\mathscr{L}_p$-spaces are in duality, i.e., for $1/p + 1/q = 1$, the dual of an $\mathscr{L}_p$-space is an $\mathscr{L}_q$-space. This important result was proved by Lindenstrauss and Rosenthal [17].

We can now give our first result.

THEOREM 1. (Stegall-Retherford [28]) *The following assertions are equivalent*:

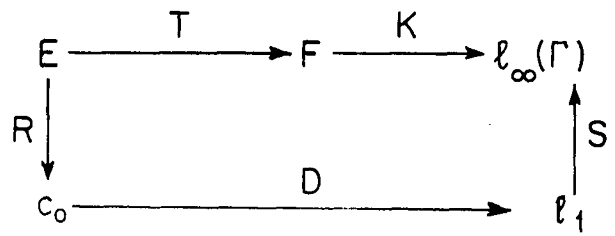
(a) $\Pi_1(E,F) = I_1(E,F)$ *for all Banach spaces* F;

(b) $QN(E,F) = N(E,F)$ *for all Banach spaces* F;

(c) $N(E,F) = FN(E,F)$ *for all Banach spaces* F; *and*,

(d) E *is an* $\mathscr{L}_\infty$*-space.*

SKETCH OF PROOF. To prove (a)$\Rightarrow$(b), let $T \in QN(E,F)$. From the remarks after the definitions, we obtain the following diagram:



where D is nuclear, K an into isometry and R and S are compact. Thus $(DR)_a$ is absolutely summing, hence, by (a), integral. Thus, if S_0 denotes the restriction of S to $\mathrm{cl}(DR(E))$, $S_0(DR)_a$ is nuclear and it follows that T is nuclear.

The implication (b) $\Rightarrow$ (c) is trivial.

The idea of the proof of (c) $\Rightarrow$ (d) is to show that (c) implies that the image of $E' \hat{\otimes} F_0$, under the canonical map J, is closed in $E' \hat{\otimes} F$ for all closed subspaces F_0 of F. Then, letting $\{F_\alpha\}$ denote the collection of all nontrivial finite dimensional subspaces of l_∞, and applying this fact to $F_0 = (\sum_{\alpha\in\Gamma} \oplus F_\alpha)_{l_\infty}$ and $F = (\sum_{\alpha\in\Gamma} \oplus l_\infty)_{l_\infty}$ one obtains, from the open mapping theorem, the existence of a constant C, so that for any α and any finite dimensional $G \supset F_\alpha$ and any $S \in \mathscr{L}(F_\alpha, E'')$, there is an $\tilde{S} \in \mathscr{L}(G, E'')$ such that $\|\tilde{S}\| \leqq C\|S\|$ and $\tilde{S}$ restricted to F_α is S. See [28, pp. 477-478] for the details. It then follows from [17] that E is an $\mathscr{L}_\infty$-space.

For (d) $\Rightarrow$ (a) one shows, using the $\mathscr{L}_\infty$-structure, that if $T \in \Pi_1(E, F)$ then the induced bilinear form T_B on $E \hat{\otimes} F' \to C$ given by

$$T_B(x, h') = \langle Tx, h' \rangle$$

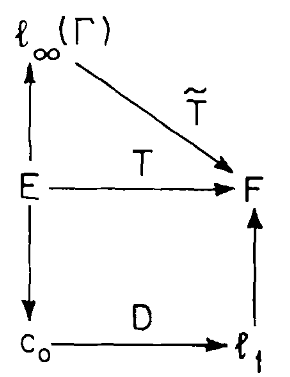
is continuous. By a result of Grothendieck [4], it follows that T is integral.

As a corollary we obtain a result of Lindenstrauss and Rosenthal [17].

COROLLARY 1. *A complemented subspace of an $\mathscr{L}_\infty$-space is an $\mathscr{L}_\infty$-space.*

PROOF. Let E be an $\mathscr{L}_\infty$-space and P a projection from E onto F. If $T \in N(F, G)$ where G is arbitrary then by Theorem 1, $TP \in FN(E, G)$. Since P is the identity on F, it follows that T is also fully nuclear. Again by Theorem 1, we obtain that F is an $\mathscr{L}_\infty$-space.

Even though the fully nuclear operators do not enjoy linear structure in general, we do obtain from Theorem 1 the following curious fact. If $T \in N(E, F)$ and if E is imbedded in a suitable $l_\infty(\Gamma)$ then it is clear from the Hahn-Banach theorem that T has a nuclear extension $\tilde{T}$. Thus we obtain the following diagram:



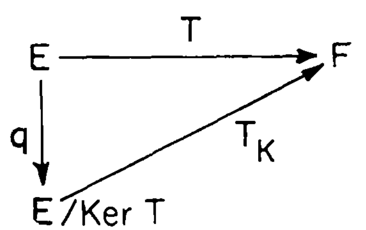
By Theorem 1, both $\tilde{T}$ and D are fully nuclear, Thus, *every nuclear operator factors through a fully nuclear operator and is the restriction of a fully nuclear operator.*

Here is another curiosity. Suppose $T: E \to F$ is compact and that $T(E)$ is infinite dimensional. Then there is an infinite dimensional subspace E_0 of E such that the restriction of T to E_0 is fully nuclear. This fact is immediate from [28, Cor. III. 10, p. 483] and recent results in [1].

Also, Theorem 1 shows that quasi-nuclear, non-nuclear operators exist in abundance.

We now present a result essentially dual to Theorem 1. For this we need the notion of a completely nuclear operator.

Consider the following diagram:



Here q is the canonical quotient map and T_K the induced operator. We say that $T \in \mathscr{L}(E,F)$ is *completely nuclear*, written $T \in CN(E,F)$, if the induced operator T_K above, is nuclear. The notion of a completely nuclear operator is dual to that of a fully nuclear operator in the following sense: *An operator $T \in FN(E,F)$ if and only if $T' \in CN(F',E')$.*

Also, if E has the nuclear adjoint property (i.e., for $T \in \mathscr{L}(E,F)$, if T' is nuclear then T must be nuclear), then $T \in CN(E,F)$ *if and only if* $T' \in FN(F',E')$.

From these remarks it is evident that the behavior of the completely nuclear operators is also pathological. For a study of this pathology see [28].

We can now give the dual to Theorem 1.

THEOREM 2. (Stegall-Retherford [28]) *The following assertions are equivalent*:

(a) *E is an $\mathscr{L}_1$-space;*

(b) *$N(F,E) = CN(F,E)$ for all Banach spaces F;*

(c) *for every Banach space F and closed subspace F_0 of F, the canonical operator*

$$F_0 \hat{\otimes} E \to F \hat{\otimes} E$$

is an isomorphism; *and*

(d) E' *is injective.*

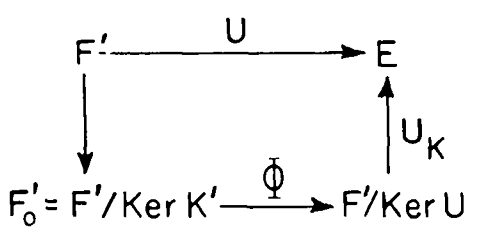
SKETCH OF PROOF. That (a) and (d) are equivalent is a result of Lindenstrauss and Rosenthal [17].

If J denotes the canonical map from $F_0 \hat{\otimes} E \to F \hat{\otimes} E$, then clearly J is an isomorphism if and only if J', which is restriction $\mathscr{L}(F,E') \to \mathscr{L}(F_0,E')$ is onto. The latter happens if and only if E' is injective. Thus (a), (c) and (d) are equivalent.

That (a) $\Rightarrow$ (b) follows from Theorem 1.

We give the proof that (b) $\Rightarrow$ (c) in more detail than in [38]. Our proof there, judging by requests for clarification, is somewhat obscure.

First one can suppose, without loss of generality, that E has the approximation property. Let I be the identity operator on E, and K the injection $F_0 \to F$. Let $U_i \in F_0 \hat{\otimes} E$ and suppose $U = \lim_i K \otimes IU_i$. Since $U_iK' = K \otimes IU_i$, U_i is 0 on $\operatorname{Ker} K'$ and so $\operatorname{Ker} K' \subset \operatorname{Ker} U$. We claim that $U = (K \otimes I)U_0$ for some $U_0 \in F_0 \hat{\otimes} E$. To see this, define $\Phi\colon F'/\operatorname{Ker} K' \to F'/\operatorname{Ker} U$ by $\Phi(f + \operatorname{Ker} K') = f + \operatorname{Ker} U$. We then have



By hypothesis, U_K is nuclear and so $U_0 = U_K\Phi$ is nuclear. But $U = (K \otimes I)U_0$ and so the range of $K \otimes I$ is closed and, since E has the approximation property, an isomorphism. Thus all four statements are equivalent.

Just as with Theorem 1, we obtain the following corollary to Theorem 2. This result was also first proved by Lindenstrauss and Rosenthal [17].

COROLLARY 2. *A complemented subspace of an $\mathscr{L}_1$-space is an $\mathscr{L}_1$-space.*

We should mention that Grothendieck [4], [6] has shown that the isometric version of Theorem 2 (c) characterizes the $L_1(\mu)$-spaces. However, there are $\mathscr{L}_1$-spaces which are not isomorphic to any $L_1(\mu)$-space. Indeed if $\alpha\colon l_1 \to L_1[0, 1]$ is a surjection, then Ker α is such a space [15], [16], [17]. Thus, Theorem 2 is an instance where both the isomorphic and isometric theory are completed.

We should also mention that Noël [19] has recently proved the results of Theorems 1 and 2 utilizing the theory of tensor products as developed by S. Chevet [2] and P. Saphar [26], [27].

In example (B), we observe that $\Pi_1(c_0, F) = N(c_0, F)$ for any Banach space F. What spaces E have this property? This question has recently been answered by Lewis and Stegall.

THEOREM 3. (Lewis-Stegall [13]) *Let E be a Banach space. Then $\Pi_1(E,F) = N(E,F)$ for all F if and only if E' is isomorphic to $l_1(\Gamma)$ for suitable Γ.*

SKETCH OF PROOF. The results of Theorem 1 and [16] imply that E' is isomorphically contained in $L_1(\mu)$ for some measure μ on Ω, and that there is a projection P of $L_1(\mu)$ onto E'. The fact that each absolutely summing operator from E to $L_1(\mu)$ must be nuclear produces a decomposition $(B_\alpha)_{\alpha \in A}$ of (Ω, μ) having the property that all of the operators

$$T_\alpha(f) = P(f\chi_{B\alpha})$$

are compact. Let $Q = Q_\alpha$ be a fixed quotient mapping from some $l^1(\Delta)$ onto E'. From [17] there is, for each $\alpha \in A$, an operator $S_\alpha: L_1(\mu) \to l_1(\Delta)$ such that $\| S_\alpha \| \leqq 2 \| P \|$ and $S_\alpha Q_\alpha = T_\alpha$. Define $S: L_1(\mu) \to l_1\,(A \times \Delta)$ by $S(f)(\alpha, \delta) = S_\alpha(f\chi_{B\alpha}(\delta)$ and define $\tilde{Q}: l_1(A \times \Delta) \to E'$ by

$$\tilde{Q}(g) = \Sigma_\alpha Q_\alpha(g(\alpha,.)).$$

Then $\tilde{Q} \cdot (S| E')$ is the identity on E' and so E' is isomorphic to a complemented subspace of $l^1(A \times \Delta)$. By Köthe's theorem [9], E' is isomorphic to some $l_1(\Gamma)$.

We give a corollary to Theorem 3 which is an excellent example of the use of operator theory to settle questions concerning the isomorphic properties of Banach spaces.

COROLLARY 3. (Stegall-Lewis [13]) *If E' is complemented in $L_1[0,1]$ then E' is isomorphic to l_1.*

PROOF. By Corollary 2, E' is an $\mathscr{L}_\infty$-space. Let $T \in \Pi_1(E,F)$. By Theorem 1, $T \in I_1(E,F)$ and since E' is separable, $T \in N(E,F)$ [4]. Thus by Theorem 3, E' is isomorphic to l_1.

We now return to example (A). It is known [16] that the $\mathscr{L}_2$-spaces are precisely the isomorphs of Hilbert space.

THEOREM 4. *The following assertions are equivalent:*

(a) *E is an $\mathscr{L}_2$-space;*

(b) *$N(F,E) = FN(F,E)$ for all Banach spaces F;*

(c) *E has the nuclear adjoint property and $CN(E,F) = N(E,F)$ for all Banach spaces F; and*

(d) *every closed subspace of E' is complemented.*

The difficult part of Theorem 4, the equivalence of (a) and (d), is the profound result of Lindenstrauss and Tzafriri [18].

To this point, we have only discussed the $\mathscr{L}_p$-spaces for $p = 1,2$ and ∞. Now every $\mathscr{L}_p$-space, $1 < p < \infty$, $p \neq 2$, is complemented in some $L_p(\mu)$ [16]. Moreover, for these values of p, a complemented subspace of $L_p(\mu)$ is either an $\mathscr{L}_p$-space or an $\mathscr{L}_2$-space [17]. Thus we now give some operator characterizations of the complemented subspaces of $L_p(\mu)$. Using these results and Theorem 4, one can, of course, give operator characterizations of the $\mathscr{L}_p$-spaces, $1 < p < \infty$, $p \neq 2$.

We need one other class of operators. An operator $T \in \mathscr{L}(E,F)$ is Cohen *p-nuclear*[3], written $T \in J_p(E,F)$, if $I \otimes T : l_p \hat{\hat{\otimes}} E \to l_p \hat{\hat{\otimes}} F$ actually maps into $l_p \hat{\otimes} F$. Here I is the identity operator on l_p.

THEOREM 5. (Cohen [3], Lewis [12], Kwapién [10]). *Let $1 < p < \infty$ and $1/p + 1/q = 1$. The following statements about the Banach space E are equivalent:*

(a) *E is isomorphic to a complemented subspace of $L_p(\mu)$ for some measure μ;*

(b) *$J_q(E,F) = I_1(E,F)$ for all Banach spaces F; and*

(c) *$T \in \Pi_q(E,F)$ implies $T' \in I_q(F',E')$ for all Banach spaces F.*

Before saying something about the proof of Theorem 5, we give one last result concerning some classes of Banach spaces which properly include the $\mathscr{L}_p$-spaces. We say, following Kwapién [10], that a Banach space E is of

S_p-type if E is isomorphic to a subspace of some L_p-(μ)space;

Q_p-type if E is isomorphic to a quotient of some $L_p(\mu)$-space; and,

SQ_p-type if E is isomorphic to a subspace of a quotient space of some $L_p(\mu)$-space.

THEOREM 6. (Kwapién [10], Holub [7]) *Let $1 \leqq p \leqq \infty$ and $1/p + 1/q = 1$. Then,*

(a) *E is of S_p-type if and only if $T \in I_q(E,F)$ implies $T' \in I_q(F', E')$ for all Banach spaces F;*

(b) *E is of Q_p-type if and only if $T \in \Pi_q(E,F)$ implies $T' \in \Pi_q(F',E')$ for all Banach spaces F; and,*

(c) *E is of SQ_p-type if and only if $T \in I_q(E,F)$ implies $T' \in \Pi_q(F',E')$ for all Banach spaces F.*

The equivalence of (a) and (b) of Theorem 5 was first proved by Lewis [12] using tensor products and a recent result of Johnson [8]. Holub [7] proved (a) of Theorem 6 using some results of Persson [21]. The remaining equivalence were proved by Kwapién [10] using the theory of Banach ideals of operators (see e.g., [25]).

In particular, Kwapién proved many interesting facts about the ideal $\Gamma_p(E, F)$ of all operators which factor through some $L_p(\mu)$-space, μ a positive measure.

We will say more about the proofs of Theorems 5 and 6 and about Banach ideals of operators in the paper which follows.

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